Zero curvature representation for classical lattice sine-Gordon equation via quantum R-matrix

A.Zabrodin*

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Abstract

Local M-operators for the classical sine-Gordon model in discrete space-time are constructed by convolution of the quantum trigonometric 4×4 R-matrix with certain vectors in its "quantum" space. Components of the vectors are identified with τ-functions of the model. This construction generalizes the known representation of M-operators in continuous time models in terms of Lax operators and classical r-matrix.

*Joint Institute of Chemical Physics, Kosygina str. 4, 117334, Moscow, Russia and ITEP, 117259, Moscow, Russia

1 Introduction

The r-matrix approach allows one to develop a unified treatment [1] of non-linear integrable equations as hamiltonian systems having enough number of conserved quantities in involution. The role of the (classical) r-matrix is to provide a universal form of Poisson brackets for elements of Lax operators.

An alternative approach [1], [2] consists in representing non-linear equations as 2D zero curvature conditions for a pair of matrix functions (called L and M operators) depending on a spectral parameter. Although this method avoids any reference to the hamiltonian aspects of the problem, the r-matrix is meaningful here, too. The alternative though less popular point of view on the r-matrix (which we are going to follow) is to treat it as a machine to produce M-operators from L-operators. Let us recall how it works.

Let $\mathcal{L}_l(z)$ be a classical 2×2 L-operator on 1D lattice with the periodic boundary condition $\mathcal{L}_{l+N}(z) = \mathcal{L}_l(z)$; z is the spectral parameter. We assume the ultralocality – Poisson brackets of dynamical variables at different sites are equal to zero. The monodromy matrix $\mathcal{T}_l(z)$ is ordered product of L-operators along the lattice from the site l to $l+N-1$:

$$\mathcal{T}_l(z) = \overleftarrow{\prod_{l+N-1\geq j\geq l}} \mathcal{L}_j(z)\,. \tag{1.1}$$

Generating function of conserved quantities is obtained by taking trace of the monodromy matrix:

$$T(z) = \mathrm{tr}\, \mathcal{T}_l(z) \tag{1.2}$$

which does not depend on l due to the periodic boundary condition. Conserved quantities obtained by expanding $\log T(z)$ in z are hamiltonians of commuting flows.

All these flows admit a zero curvature representation. The generating function of corresponding M-operators is given by [4], [1]

$$M_l(z;w) = T^{-1}(w)\,\mathrm{tr}_1\left[r(\frac{z}{w})(\mathcal{T}_l(w)\otimes I)\right], \tag{1.3}$$

where I is the unity matrix, $r(z)$ is the (classical) 4×4 r-matrix. It acts in the tensor product of two 2-dimensional spaces and, therefore, has natural block structure. In (1.3), tr_1 means trace in the first space. Expanding the r.h.s. of eq. (1.3) in w, one gets M-operators depending on the spectral parameter z. From the hamiltonian point of view, the zero curvature condition follows from r-matrix Poisson brackets for elements of the L-operator. In general, $M_l(z;w)$ is a non-local quantity.

A way to construct local M-operators from the generating function is well known [5], [1], [3]. Suppose there exists a value z_0 of the spectral parameter such that the L-operator degenerates for all l, i.e. $\det \mathcal{L}_l(z_0) = 0$. This means that $\mathcal{L}_l(z_0)$ is a 1-dimensional projector:

$$\mathcal{L}_l(z_0) = \frac{\big|\alpha_l\big\rangle\big\langle\beta_l\big|}{\lambda_l}\,, \tag{1.4}$$

where

$$\big|\alpha\big\rangle = \begin{pmatrix}\alpha^{(1)}\\ \alpha^{(2)}\end{pmatrix}, \qquad \big\langle\beta\big| = \big(\beta^{(1)},\, \beta^{(2)}\big) \tag{1.5}$$

are two-component vectors and λ_l is a scalar normalization factor. (Of course this factor can be included in the vectors.) Components of the vectors as well as the λ_l depend on dynamical variables. It is easy to see that $M_l(z;z_0)$ is a local quantity:

$$M_l(z) \equiv M_l(z;z_0) = \frac{\big\langle\beta_l\big|r(z/z_0)\big|\alpha_{l-1}\big\rangle}{\big\langle\beta_l\big|\alpha_{l-1}\big\rangle} \tag{1.6}$$

(note that the normalization factor cancels). The scalar product is taken in the first space only, so the result is a 2×2 matrix with the spectral parameter z. It generates infinitezimal continuous time shifts. The zero curvature condition giving rise to non-linear integrable equations reads

$$\partial_t \mathcal{L}_l(z) = M_{l+1}(z)\mathcal{L}_l(z) - \mathcal{L}_l(z)M_l(z)\,. \tag{1.7}$$

The goal of this work is to extend eq. (1.6) to fully discretized analogues of classical 2D integrable models and to obtain a similar representation for M-operators corresponding to *discrete time flows*. By the fully discretized models we mean a family of 2D partial difference integrable equations introduced and studied by Hirota [6]-[8]: discrete analogues of the KdV, the Toda chain, the sine-Gordon (SG) equations, etc. Our guiding principle is Miwa's interpretation [9] of the discrete time flows, in which discretized integrable equations are treated as members of the same infinite hierarchy as their continuous counterparts. (This idea was further developed in [10] as a general method to produce discrete soliton equations from continuous ones.) Although in this paper we deal with the discrete SG equation only, there is no doubt that the results can be more or less straightforwardly extended to other integrable models on 2D lattice.

Let us outline the results. A formula for discrete M-operators of the type (1.6) does exist. Remarkably, the structure of the formula remains the same, with the classical r-matrix being substituted by quantum R-matrix. Specifically, we show that the following representation of discrete M-operators holds:

$$\mathcal{M}_l(z) = \frac{\big\langle \beta_l \big| R(z/z_0) \big| \check{\beta}_{l-1} \big\rangle}{\big\langle \beta_l \big| \alpha_{l-1} \big\rangle}\,, \qquad \big|\check{\beta}_l\big\rangle \equiv \sigma_1 \big|\beta_l\big\rangle \tag{1.8}$$

(here and below $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices). In the r.h.s., $R(z)$ is a *quantum* 4×4 R-matrix to be specified below with the "quantum" parameter q related to the discrete time lattice spacing. We stress that vectors $\big|\alpha_l\big\rangle$ and $\big|\beta_l\big\rangle$ in this formula are *the same* as in eq. (1.6).

The M-operator (1.8) generates shifts of a discrete time variable m. The discrete zero curvature condition

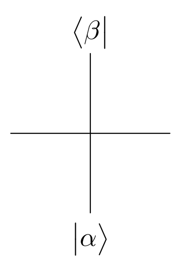
$$\mathcal{M}_{l+1,m}(z)\mathcal{L}_{l,m}(z) = \mathcal{L}_{l,m+1}(z)\mathcal{M}_{l,m}(z) \tag{1.9}$$

(with the same L-operator as before) gives rise to non-linear integrable equations in discrete space-time listed above. Under the condition that these equations are satisfied a similar R-matrix representation for the L-operator itself is valid:

$$\mathcal{L}_l(z) = \frac{\big\langle \beta_l \big| R^{(-)}(z/z_0) \big| \alpha_l \big\rangle}{\big\langle \beta_l \big| \alpha_{l-1} \big\rangle}\,. \tag{1.10}$$

Here $R^{(-)}(z)$ is another quantum R-matrix which differs from $R(z)$ by Drinfeld's twist.

As soon as the quantum R-matrix comes into play, we can use customary conventions of the algebraic Bethe ansatz approach [11] ,[3] to represent formulas (1.8), (1.10) in the figure:

$\big\langle \beta \big| R(z) \big| \alpha \big\rangle =$ 

The scalar product is taken in the "quantum" (vertical) space of the R-matrix, so the result is a 2×2 matrix in its auxiliary (horizontal) space.

The change of dynamical variables to the pair of vectors $\big|\alpha_l\big\rangle$, $\big|\beta_l\big\rangle$ plays a key role in the construction. In the papers [5] on exact lattice regularization of integrable models components of these vectors were expresed in terms of canonical variables of the model. Those formulas looked complicated and were hardly considered as something instructive. Here we add a remark which may help to understand their meaning. Taking into account equations of motion of the completely discretized model derived from the discrete zero curvature condition (1.9), we show that (suitably normalized) components of the vectors $\big|\alpha_l\big\rangle$, $\big|\beta_l\big\rangle$ are *τ-functions*. The τ-function is known to be one of the most fundamental objects of the theory (see e.g. [12]).

In this paper we explain the construction for the simplest example – the lattice SG model. In the literature, there are two lattice versions of the classical SG model: Hirota's discrete analogue of the SG

equation on a space-time lattice [8] and the model on a space lattice with continuous time introduced by Izergin and Korepin [5] and further studied in [13]. Actually, they are closely connected with each other and, furthermore, each of the two models can be better understood with the help of the other one. They have common L-operator (which can be represented in the form (1.10)). The M-operators are different: in the latter case the r.h.s. of eq. (1.6) with the classical trigonometric r-matrix generates continuous time derivatives while in the former case discrete time shifts are generated by (1.8). The R-matrix entering the r.h.s. of this formula is in this case the simplest trigonometric solution of the Yang-Baxter equation (the R-matrix of the XXZ spin chain). At last, we demonstrate how eq. (1.6) is reproduced from eq. (1.8) in the continuous time limit.

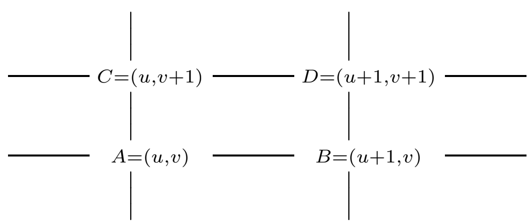
2 Lattice SG models with discrete and continuous time

By the discrete SG model on the space-time lattice we mean the Faddeev-Volkov version [14] of Hirota's discrete analogue [8] of the classical SG equation:

$$\begin{aligned} &\nu\psi(u,v+1)\psi(u+1,v+1) - \nu\psi(u,v)\psi(u+1,v) \\ = \quad &\mu\psi(u+1,v)\psi(u+1,v+1) - \mu\psi(u,v)\psi(u,v+1)\,. \end{aligned} \tag{2.1}$$

This is a non-linear partial difference equation for a function ψ of the two variables u, v, where μ, ν are constant parameters. We call u, v *chiral space-time coordinates*. It was shown [14], [15] that eq. (2.1) contains both SG and KdV equations as results of different continuum limits.

Let



be an elementary cell of the u, v-lattice; in this notation eq. (2.1) reads

$$\psi_C\psi_D - \psi_A\psi_B = \frac{\mu}{\nu}\big(\psi_B\psi_D - \psi_A\psi_C\big)\,. \tag{2.2}$$

The non-linear equation (2.2) can be represented [14] as a discrete zero curvature condition. Set

$$\phi(u,v) \equiv \psi^{\frac{1}{2}}(u,v) \tag{2.3}$$

and consider the chiral L-operator [14], [16]

$$L_{B\leftarrow A}(z;\mu) = \begin{pmatrix} \mu\phi_B\phi_A^{-1} & z\phi_B^{-1}\phi_A^{-1} \\ z\phi_B\phi_A & \mu\phi_B^{-1}\phi_A \end{pmatrix}. \tag{2.4}$$

Equation (2.2) is equivalent to the zero curvature condition

$$L_{D\leftarrow B}(z;\nu)L_{B\leftarrow A}(z;\mu) = L_{D\leftarrow C}(z;\mu)L_{C\leftarrow A}(z;\nu) \tag{2.5}$$

on the chiral space-time lattice.

The chiral L-operators are building blocks for more complicated ones. For our purpose we need to pass from chiral to the "direct" space-time coordinates $l = \frac{1}{2}(u+v)$, $m = \frac{1}{2}(u-v)$. (From now on we refer to l (resp., m) as discrete space (resp., discrete time) coordinate.) To do that, consider "composite" operators which generate shifts along the diagonals $A \to D$ and $C \to B$ respectively:

$$\hat{\mathcal{L}}_{D\leftarrow A}(z) = z^{-1}L_{D\leftarrow C}(z;\mu)L_{C\leftarrow A}(z;\nu), \tag{2.6}$$

$$\hat{\mathcal{M}}_{B\leftarrow C}(z) = z^{-1}(z^2-\nu^2)L_{B\leftarrow A}(z;\mu)\,[L_{C\leftarrow A}(z;\nu)]^{-1}\,. \tag{2.7}$$

Using (2.4), we have:

$$\hat{\mathcal{L}}_{D\leftarrow A}(\mu z) = \begin{pmatrix} \mu\frac{\phi_A}{\phi_D}z+\nu\frac{\phi_D}{\phi_A}z^{-1} & \phi_C^{-2}\left(\mu\frac{\phi_D}{\phi_A}+\nu\frac{\phi_A}{\phi_D}\right) \\ \phi_C^2\left(\mu\frac{\phi_A}{\phi_D}+\nu\frac{\phi_D}{\phi_A}\right) & \mu\frac{\phi_D}{\phi_A}z+\nu\frac{\phi_A}{\phi_D}z^{-1} \end{pmatrix}, \tag{2.8}$$

$$\hat{\mathcal{M}}_{B\leftarrow C}(\mu z) = \begin{pmatrix} \mu\frac{\phi_C}{\phi_B}z-\nu\frac{\phi_B}{\phi_C}z^{-1} & \phi_A^{-2}\left(\mu\frac{\phi_B}{\phi_C}-\nu\frac{\phi_C}{\phi_B}\right) \\ \phi_A^2\left(\mu\frac{\phi_C}{\phi_B}-\nu\frac{\phi_B}{\phi_C}\right) & \mu\frac{\phi_B}{\phi_C}z-\nu\frac{\phi_C}{\phi_B}z^{-1} \end{pmatrix}. \tag{2.9}$$

Discrete zero curvature condition for these operators has the form (1.9). It gives rise to a system of non-linear equations for fields at the vertices of four neighbouring elementary cells, which is a direct corollary of the basic equation (2.1).

Let us turn to the lattice SG model with continuous time introduced by Izergin and Korepin in [5]. Its L-operator on l-th site has the form[1]

$$\hat{\mathcal{L}}_l^{(IK)}(z) = \begin{pmatrix} z\chi_l+z^{-1}\chi_l^{-1} & s^{-\frac{1}{2}}\varphi_l\pi_l \\ s^{-\frac{1}{2}}\varphi_l\pi_l^{-1} & z\chi_l^{-1}+z^{-1}\chi_l \end{pmatrix}. \tag{2.10}$$

Here π_l, χ_l are exponentiated canonical variables on the lattice, $\varphi_l = \left[1+s(\chi_l^2+\chi_l^{-2})\right]^{\frac{1}{2}}$ and s is a parameter of the model.

As is known, discretizing time in integrable models consists merely in changing the M-operator and leaving the L-operator unchanged, so we are to identify the L-operators (2.10) and (2.8). For that purpose, consider composite fields

$$\pi(u,v) = [\psi(u+1,v)\psi(u,v+1)]^{\frac{1}{2}}\,, \qquad \chi(u,v) = \left[\frac{\psi(u,v)}{\psi(u+1,v+1)}\right]^{\frac{1}{2}} \tag{2.11}$$

on the chiral lattice and set $\pi_l = \pi(l,l)$, $\chi_l = \chi(l,l)$ at the constant time slice $m=0$. Finally, identifying $s = \mu\nu(\mu^2+\nu^2)^{-1}$ and taking into account the equation of motion (2.1), we conclude that

$$\hat{\mathcal{L}}_l^{(IK)}(z) = (\mu\nu)^{-\frac{1}{2}}\hat{\mathcal{L}}_l\big((\mu\nu)^{\frac{1}{2}}z\big)\,. \tag{2.12}$$

Here we use the natural notation

$$\hat{\mathcal{L}}_l(z) \equiv \hat{\mathcal{L}}_{D_l\leftarrow A_l}(z)\,, \tag{2.13}$$

where $A_l = (l,l)$, $D_l = (l+1,l+1)$ (note that in this notation $D_l = A_{l+1}$). (In what follows we use the similar notation for the M-operator (2.9): $\hat{\mathcal{M}}_{\bar{B}_l\leftarrow A_l}(z) \equiv \hat{\mathcal{M}}_l(z)$, where $\bar{B}_l = (l+1,l-1)$.)

The L-operator $\hat{\mathcal{L}}_l^{(IK)}(z)$ has two degeneracy points $z_0^\pm$. In terms of the parameters μ,ν they are $z_0^\pm = (\mu/\nu)^{\pm\frac{1}{2}}$. Using (2.12) and (2.8), it is not difficult to represent $\hat{\mathcal{L}}_l^{(IK)}(z_0^\pm)$ in the form (1.4) with the r.h.s. expressed through the field $\psi(u,v)$.

3 Hirota's bilinear formalism

The idea of Hirota's approach is to treat eq. (2.1) as a consequence of 3-term bilinear equations for τ-functions, which are simpler and in a sense more fundamental. In this section we give a minimal extraction from Hirota's method necessary for what follows (for more details see e.g. [10], [17]).

[1] We use a slightly modified form of the L-operator originally suggested in [5]. They differ by a unitary transformation and multiplication by the matrix σ_2 from the left. The latter modification takes into account the fact that we deal with the Faddeev-Volkov equation (2.1) rather than Hirota's discrete SG equation itself.

In the case at hand we need two τ-functions: τ and $\hat{\tau}$. Making the substitution

$$\psi(u,v) = \frac{\hat{\tau}(u,v)}{\tau(u,v)}, \tag{3.1}$$

we immediately see that eq. (2.1) follows from the couple of Hirota's bilinear equations [8]

$$\begin{aligned} (\nu-\mu)\hat{\tau}_A\tau_D &= \nu\tau_B\hat{\tau}_C - \mu\hat{\tau}_B\tau_C\,, \\ (\nu-\mu)\tau_A\hat{\tau}_D &= \nu\hat{\tau}_B\tau_C - \mu\tau_B\hat{\tau}_C\,. \end{aligned} \tag{3.2}$$

The equivalent form of these equations,

$$\begin{aligned} (\nu+\mu)\tau_B\hat{\tau}_C &= \mu\tau_A\hat{\tau}_D + \nu\hat{\tau}_A\tau_D\,, \\ (\nu+\mu)\hat{\tau}_B\tau_C &= \mu\hat{\tau}_A\tau_D + \nu\tau_A\hat{\tau}_D\,, \end{aligned} \tag{3.3}$$

is equally useful in what follows. At last, we point out the relation

$$\tau(u-1,v)\hat{\tau}(u+1,v) + \hat{\tau}(u-1,v)\tau(u+1,v) = 2\tau(u,v)\hat{\tau}(u,v) \tag{3.4}$$

valid for the τ-functions of the discrete SG model.

Let us give some additional remarks. Equations (3.2) are to be thought of as being embedded into the infinite 2D Toda lattice hierarchy with 2-periodic reduction [18]. This embedding implies that parameters μ and ν are Miwa's variables [9] corresponding to the two elementary discrete flows u, v. Miwa's variables play the role of inverse lattice spacings in the chiral directions. Lattice spacing in the m-direction is then to be identified with $\frac{\mu-\nu}{\mu\nu}$. One should note, however, that in this approach the chiral coordinate axes are in general not orthogonal to each other. In particular, as it is seen from eqs. (3.2), at $\mu = \nu$ one must *identify* u and v, so that the 2D lattice collapses to a 1D one. In this sense eq. (3.4) follows from (3.3) at $\nu = \mu$.

4 Quantum R-matrix in the classical lattice SG model

Our goal is to represent the L and M operators (2.8), (2.9) as convolutions of quantum R-matrix with some vectors in the "quantum" space.

Consider quantum R-matrices of the form

$$R^{(\pm)}(z) = \begin{pmatrix} a(z) & 0 & 0 & 0 \\ 0 & \pm b(z) & c(z) & 0 \\ 0 & c(z) & \pm b(z) & 0 \\ 0 & 0 & 0 & a(z) \end{pmatrix}, \tag{4.1}$$

where

$$a(z) = qz - q^{-1}z^{-1}, \quad b(z) = z - z^{-1}, \quad c(z) = q - q^{-1}. \tag{4.2}$$

Here q is a "quantum" parameter and z is the spectral parameter. The R-matrices $R^{(+)}$ and $R^{(-)}$ differ by Drinfeld's twist. Both of them satisfy the quantum Yang-Baxter equation. In the Introduction we have used the notation $R^{(+)}(z) = R(z)$. When necessary, we write $R(z) = R(z;q)$.

The R-matrix has 2×2 block structure with respect to the "auxiliary" space. Let i, i' number block rows and columns and let j, j' number rows and columns inside each block (i.e., in the "quantum" space). Then matrix elements of the R-matrix of the form (4.1) are denoted by $R(z)^{ii'}_{jj'}$. Let $|\alpha\rangle$, $|\beta\rangle$ be two vectors in the quantum space (see (1.5)). Each block of the R-matrix is an operator in the quantum space. Consider its action to $|\alpha\rangle$ and subsequent scalar product with $\langle\beta|$ (see the figure in the Introduction). The result is a 2×2 matrix in the auxiliary space:

$$\langle\beta|R(z)|\alpha\rangle_{ii'} = \sum_{jj'} R(z)^{ii'}_{jj'}\alpha^{(j')}\beta^{(j)}.$$

Substituting the matrices (4.1), we find:

$$\big\langle \beta \big| R^{(\pm)}(z) \big| \alpha \big\rangle = \left(\begin{array}{cc} \beta^{(1)}\alpha^{(1)}a(z) \pm \beta^{(2)}\alpha^{(2)}b(z) & \beta^{(2)}\alpha^{(1)}c(z) \\ \beta^{(1)}\alpha^{(2)}c(z) & \pm\beta^{(1)}\alpha^{(1)}b(z) + \beta^{(2)}\alpha^{(2)}a(z) \end{array} \right). \tag{4.3}$$

We are going to identify r.h.s. of eqs. (2.8), (2.9) with this matrix. The key step is to express matrix elements of the L and M operators through the τ-functions according to (3.1) and after that make use of Hirota's bilinear equations (3.2), (3.3) when necessary. The best result is achieved after the diagonal gauge transformation

$$\hat{\mathcal{L}}_{D\leftarrow A}(z) \longrightarrow \mathcal{L}_{A\leftarrow D}(z) = \left(\frac{\tau_D \hat{\tau}_D}{\tau_A \hat{\tau}_A} \right)^{\frac{1}{2}} \hat{\mathcal{L}}_{D\leftarrow A}(z)\,, \tag{4.4}$$

$$\hat{\mathcal{M}}_{B\leftarrow C}(z) \longrightarrow \mathcal{M}_{B\leftarrow C}(z) = \left(\frac{\tau_B \hat{\tau}_B}{\tau_C \hat{\tau}_C} \right)^{\frac{1}{2}} \hat{\mathcal{M}}_{B\leftarrow C}(z)\,. \tag{4.5}$$

Omitting details of the computation, we present the final result. Set

$$\big|\alpha\big\rangle = \left(\begin{array}{c} \tau \\ \hat{\tau} \end{array} \right), \qquad \big|\beta\big\rangle = \left(\begin{array}{c} \hat{\tau} \\ \tau \end{array} \right), \qquad q = \frac{\mu}{\nu}\,. \tag{4.6}$$

Using eqs. (3.2), (3.3), we get:

$$\big\langle \beta_B \big| R^{(-)}(z) \big| \alpha_C \big\rangle = \frac{\mu - \nu}{\mu\nu} \tau_A \hat{\tau}_A \mathcal{L}_{D\leftarrow A}(\mu z)\,, \tag{4.7}$$

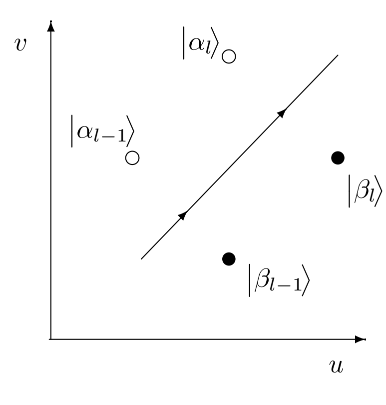
$$\big\langle \beta_D \big| R^{(+)}(z) \big| \alpha_A \big\rangle = \frac{\mu + \nu}{\mu\nu} \tau_C \hat{\tau}_C \mathcal{M}_{B\leftarrow C}(\mu z)\,. \tag{4.8}$$

Finally, let us pass to the discrete space-time coordinates l, m and consider the slice of the lattice at $m = 0$. The notations of the type (2.13) explained in Sect.2 are again convenient. Making use of eq. (3.4), we arrive at the formulas

$$\mathcal{L}_l(\mu z) = \frac{2\mu\nu}{\mu - \nu} \, \frac{\big\langle \beta_l \big| R^{(-)}(z) \big| \alpha_l \big\rangle}{\big\langle \beta_l \big| \alpha_{l-1} \big\rangle}\,, \tag{4.9}$$

$$\mathcal{M}_l(\mu z) = \frac{2\mu\nu}{\mu + \nu} \, \frac{\big\langle \beta_l \big| R^{(+)}(z) \big| \check{\beta}_{l-1} \big\rangle}{\big\langle \beta_l \big| \alpha_{l-1} \big\rangle}\,, \tag{4.10}$$

which up to the constant prefactors coincide with the ones announced in the Introduction. Location of the vectors $|\alpha_l\rangle$, $|\beta_l\rangle$ on the lattice with respect to the $m = 0$ slice is shown in the figure:



Explicitly, they are:

$$\big|\alpha_l\big\rangle = \left(\begin{array}{c} \tau(l, l+1) \\ \hat{\tau}(l, l+1) \end{array} \right), \qquad \big|\beta_l\big\rangle = \left(\begin{array}{c} \hat{\tau}(l+1, l) \\ \tau(l+1, l) \end{array} \right). \tag{4.11}$$

The normalization factor in eq. (1.4) is equal to $\lambda_l = \mu\nu(\mu - \nu)^{-1}\tau(l, l)\hat{\tau}(l, l)$.

5 On the continuous time limit

In this section we show that the r-matrix formula for local M-operators (1.6) is a degenerate case of eq. (1.8). As it was already mentioned at the end of Sect. 3, $\frac{\mu-\nu}{\mu\nu}$ plays the role of lattice spacing of the discrete time variable m. A naive continuous time limit would then be $\nu \to \mu$ that means $q \to 1$ in the R-matrix, so the first non-trivial term in its expansion in $q-1$ is just the classical r-matrix. This agrees with eq. (1.6). However, this limiting procedure would imply $\lim_{q\to 1} \left|\check{\beta}_l\right\rangle = \left|\alpha_l\right\rangle$ that is certainly wrong in general (see e.g. the figure in the previous section). The correct limiting transition to a continuous time coordinate needs some clarification.

The limit $\nu \to \mu$ changes parameters of the space lattice and the L-operator. That is the reason why the naive limit does not work. In the correct limiting procedure, the time lattice spacing must approach zero independently of μ and ν.

To achieve the goal, we introduce v' – another "copy" of the chiral flow v with the parameter ν', so now we have a 3D lattice. Equations (3.2) are valid in each 2D section of this lattice of the form $v' = \text{const}$, $u = \text{const}$ or $v = \text{const}$. For instance, in the latter case we have (cf. (3.2)):

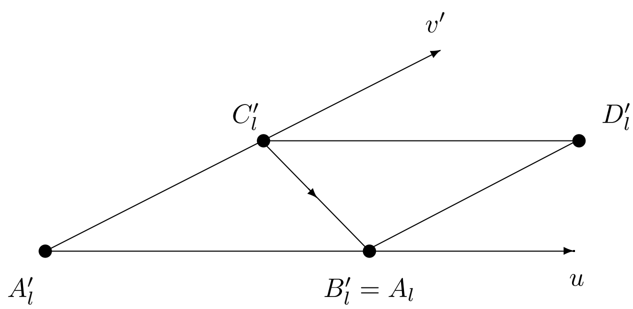
$$(\nu'-\mu)\hat{\tau}(u,v,v')\tau(u+1,v,v'+1) = \nu'\tau(u+1,v,v')\hat{\tau}(u,v,v'+1) - \mu\hat{\tau}(u+1,v,v')\tau(u,v,v'+1)\,,$$
$$(\nu'-\mu)\tau(u,v,v')\hat{\tau}(u+1,v,v'+1) = \nu'\hat{\tau}(u+1,v,v')\tau(u,v,v'+1) - \mu\tau(u+1,v,v')\hat{\tau}(u,v,v'+1)\,. \tag{5.1}$$

Now we can tend $\nu' \to \mu$ leaving ν unchanged. Set

$$q' = \frac{\mu}{\nu'} = 1 + \varepsilon + O(\varepsilon^2)\,, \qquad \varepsilon \to 0\,. \tag{5.2}$$

The small parameter ε has the meaning of lattice spacing in the auxiliary direction $m' = \frac{1}{2}(u - v')$.

Discrete M-operators are defined up to multiplication by a scalar function of z independent of dynamical variables. To pursue the continuous time limit, it is convenient to normalize the M-operator in such a way that $\mathcal{M}_l(z) = I$ at $\varepsilon = 0$. Then the next term (of order ε) yields a local M-operator of continuous time flow. To find such an M-operator at the l-th site of the 1D lattice (diagonally embedded into the u,v-lattice as (l,l)), we should expand the discrete M-operator $\mathcal{M}_{B'_l \leftarrow C'_l}(z)$ which generates the shift $(l-1,l,1) \longrightarrow (l,l,0)$ on the 3D lattice with coordinates (u,v,v'). This is illustrated by the figure



which shows the u,v'-section of the 3D lattice spanned by u,v,v'. Coordinates of the parallelogram vertices are: $A'_l = (l-1,l,0)$, $B'_l = A_l = (l,l,0)$, $C'_l = (l-1,l,1)$, $D'_l = (l,l,1)$. The point C'_l tends to the point $B'_l = A_l$ as $\nu' \to \mu$, so the parallelogram collapses to the u-axis. We have:

$$\mathcal{M}_{B'_l \leftarrow C'_l}(z) = I + \varepsilon M_l(z) + O(\varepsilon^2)\,, \tag{5.3}$$

where

$$M_l(\mu z) = \frac{1}{z - z^{-1}} \begin{pmatrix} \frac{1}{2}(z+z^{-1})\frac{\tau(l-1,l)\hat{\tau}(l+1,l)}{\tau(l,l)\hat{\tau}(l,l)} & \frac{\tau(l-1,l)\tau(l+1,l)}{\tau(l,l)\hat{\tau}(l,l)} \\ \frac{\hat{\tau}(l-1,l)\hat{\tau}(l+1,l)}{\tau(l,l)\hat{\tau}(l,l)} & \frac{1}{2}(z+z^{-1})\frac{\hat{\tau}(l-1,l)\tau(l+1,l)}{\tau(l,l)\hat{\tau}(l,l)} \end{pmatrix}. \tag{5.4}$$

By definition, the classical r-matrix is

$$
\begin{aligned}
r(z) &= \lim_{\varepsilon\to 0} \frac{R^{(+)}(z;q') - (z-z^{-1}) I\otimes I}{\varepsilon(z-z^{-1})} \\
&= \frac{1}{2(z-z^{-1})} \left[(z+z^{-1}) I\otimes I + 2\sigma_1\otimes\sigma_1 + 2\sigma_2\otimes\sigma_2 + (z+z^{-1})\sigma_3\otimes\sigma_3 \right] .
\end{aligned} \tag{5.5}
$$

Comparing with (5.4), we obtain the r-matrix representation

$$
M_l(\mu z) = \frac{\langle \beta_l | r(z) | \alpha_{l-1}\rangle}{\langle \beta_l | \alpha_{l-1}\rangle} \tag{5.6}
$$

of the M-operator (see (1.6)) with the trigonometric classical r-matrix (5.5).

6 Conclusion

The main result of this work is the R-matrix representation (4.9), (4.10) of the local L-M pair for the classical lattice SG model with discrete space-time coordinates. In our opinion, the very fact that the typical quantum R-matrix naturally arises in a purely classical problem is important and interesting by itself. As a by-product, we have shown that components of the vectors $|\alpha\rangle$, $|\beta\rangle$ (representing the L-operator at the degeneracy point, see (1.4)) are τ-functions.

Let us recall that the quantum Yang-Baxter equation already appeared in connection with purely classical problems, though in a different context [19], [20]. However, the class of solutions relevant to classical problems is most likely very far from R-matrices of known quantum integrable models. In our construction, the role of the quantum Yang-Baxter equation remains obscure; instead, the most popular 4×4 trigonometric quantum R-matrix is shown to take part in the zero curvature representation of the classical discrete SG model. We believe that a conceptual explanation of this phenomenon nevertheless relies on the Yang-Baxter equation.

We should stress that the "quantum deformation parameter" q of the trigonometric R-matrix in our context seems to have nothing to do with any kind of quantization. Indeed, it is related to the mass parameter and the lattice spacing of the classical model. This fact suggests to ask for a non-standard hidden R-matrix structure of the model, which might survive and show up on the quantum level, too. Another interesting question is to find a discrete time analogue of the non-local generating function (1.3) of M-operators.

At last, let us point out that the R-matrix representation can be extended to M-operators of a more general model – partially anisotropic classical Heisenberg spin chain in discrete time. This system includes the discrete KdV and some other equations as particular cases. The results will be reported elsewhere.

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